Pergamon

Computers Math. Applic. Vol. 31, No. 12, pp. 49-59, 1996

0898-1221/96 $15.00 + 0.00

S0898-1221(96)00075-2

Computer Aided Proof for the Global Stability of Lotka-Volterra Systems

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(Received November 1994; revised and accepted February 1996)

Abstract—By modifying the mechanical method of determining LaSalle's invariant sets for Lotka-Volterra chain systems [1,2] and repeatedly using Wu's characteristic set method [3,4], it is proved that for a class of Lotka-Volterra loop systems, the locally asymptotically stable positive equilibrium point must be globally stable.

Keywords—Lotka-Volterra system, Global stability, Wu method, Symbolic computation.

1. INTRODUCTION

Since the stability theory was established by Liapunov one hundred years ago, Liapunov methods have been the main tools in the study of stability of solutions for differential equations. The Liapunov second method is the major mathematical method for the analysis of the stability, especially the global stability.

In the initial stage, Liapunov stated his second method only in the sense of local stability. In 1952, Barbashin and Krasovskii [5] extended Liapunov's second method to the case of global stability. And LaSalle [6] developed this method further to obtain the extended Liapunov stability theorem referred as to LaSalle's invariance principle which contains the usual Liapunov-like theorems on stability and instability of systems of differential equations. In accordance with this principle, if the structure of the LaSalle's invariant set is determined in the discussion of Liapunov stability, then one knows whether solutions of the system are stable or not. In many cases, for a given system, the determination of LaSalle's invariant set is easy provided a Liapunov function is found.

Chen [7] proposed a conjecture of determining the LaSalle's invariant sets for general n-dimensional Lotka-Volterra chain systems. It is known that the conjecture is trivial for $n \le 4$ (see [8]) and had been proved by the author to be true for $n = 5, 6$ (see [9,10]). Nevertheless, the proof contains certain specific techniques and does not seem to applicable for higher dimensional systems.

One knows that the key to prove the conjecture is to establish the necessary and sufficient conditions for the LaSalle's invariant set of a given system to contain nonconstant solutions. Liu *et al.* [1] first transformed the stability problem of a family of Lotka-Volterra systems to be one of system of polynomial equations, then employed the basic principle, namely, Wu's well-ordering principle, of characteristic sets method [3,4] to drive these conditions again in a mechanical way with computers. Further, adopting the proposed mechanical method, they [2]

Project supported by the Foundation of the State Education Commission of China.
The author is very grateful to L. S. Chen and the referee for their suggestions.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX determined the LaSalle's invariant sets for seven-dimensional Lotka-Volterra food chain systems and showed that Chen's conjecture is true for these systems. Precisely, in [1,2], by using Wu's method, they could obtain directly the characteristic set of polynomials from a given set of polynomials. Then from this characteristic set, the sufficient and necessary conditions for the systems to be globally stable were obtained.

Although their approach is applicable for higher dimensional or more complicated systems, theoretically, the polynomials in these problems become too large to manipulate. Namely, we cannot obtain directly the characteristic set from a given set of polynomials in the sense of Wu. In this paper, based on the specific form of the systems, we modify the method proposed by Liu *et al.* [1,2] and apply it to some Lotka-Volterra cycle systems in which the problem for manipulating polynomials is much more complicated than those solved in [1,2].

Section 2 contains the basic results about LaSalle's invariant sets for Lotka-Volterra systems; the characteristic set method and the application of this method to our problems are illustrated in Section 3; in Section 4, global stability results for a certain class of Lotka-Volterra cycle systems are proved based on the structure of the LaSalle's invariant sets of the systems, and in Section 5, the modified manipulation method together with computers is applied to the proof of our main result for the structure of these sets.

2. BACKGROUND CONCEPTS AND RESULTS

Consider the following general n-dimensional Lotka-Volterra system:

$$\dot{x}_i = x_i \sum_{j=1}^{n} a_{ij}\left(x_j - x_j^*\right), \qquad i = 1, 2, \dots, n. \tag{1}$$

Here $x^* = (x_1^*, x_2^*, \dots, x_n^*)^\top$ is a unique positive equilibrium point of system (1). Then the Volterra's Liapunov function [11]

$$V(x) = \sum_{i=1}^{n} c_i \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}\right)$$

has a time derivative along the solutions of system (1) given by

$$\dot{V}(x) = \frac{1}{2}\left(x - x^*\right)^\top \left(CA + A^\top C\right)\left(x - x^*\right),$$

where $A = (a_{ij})_{n\times n}$ is the species interaction matrix of (1) and $C = \operatorname{diag}(c_i)$. For the interaction matrix A of system (1), the following class is important when we consider the global stability of x^*.

DEFINITION 1.

(i) *Matrix A is Volterra-Liapunov stable, if there exists a positive diagonal matrix C such that $CA + A^\top C$ is negative definite;*

(ii) *A is Volterra-Liapunov semistable, if $CA + A^\top C$ is negative semidefinite.*

Based on Liapunov global stability theorem, we have the following lemma.

LEMMA 1. *(See [11].) If A is Volterra-Liapunov stable, then x^* is globally stable in* $\operatorname{int} R_+^n = \{x \in R^n \mid x_i > 0, i = 1, \dots, n\}$.

After combining Lemma 1 with LaSalle's extended stability theorem [6], the following Lemma 2 was obtained by Harrison [12], Hsu [13] and Krikorian [14].

LEMMA 2. *If A is Volterra-Liapunov semistable, then every solution of system (1) in* $\operatorname{int} R_+^n$ *tends to the maximal invariant set M contained in the following set:*

$$E = \left\{x \in \operatorname{int} R_+^n \mid \left(x - x^*\right)^\top \left(CA + A^\top C\right)\left(x - x^*\right) = 0\right\}. \tag{2}$$

Here M is called a LaSalle's invariant set of the system.

It is known that the positive equilibrium point x^* is globally stable if $M = \{x^*\}$. Hereafter, we call system (1) globally stable if and only if x^* is globally stable in int R^n_+.

By the Liapunov stability theorem, we know that each solution in M is bounded. According to Lemma 2, if $x(t)$ is a solution belonging to M, since M is invariant and $\dot{V}(x(t)) = 0$, then there is a constant $c \geq 0$ such that $V(x(t)) = c$, for all $t \geq 0$. The Hessian of V being positive definite implies that either $x(t) = x^*$ or $x(t)$ is a bounded solution which does not tend to x^*. Namely, we have shown Lemma 3.

LEMMA 3. *If $x(t)$ is a solution in M, then either $x(t) = x^*$ or $x(t)$ is a bounded and strictly positive solution which does not tend to x^*.*

Since, in this paper, we will discuss prey-predator system, the following assumptions are made (H):

(a) $a_{ii} \leq 0$ $(i = 1, 2, \dots, n)$ with at least one $a_{ii} < 0$;
(b) $a_{ij}a_{ji} \leq 0$, $a_{ij} = 0$ iff $a_{ji} = 0$ $(i \neq j;\ i, j = 1, \dots, n)$.

DEFINITION 2. *[9] System (1) is called a cycle system if there are distinct indices $i_1, i_2, \dots, i_m$ $(m \geq 3)$ such that none of the elements $a_{i_1i_2}, a_{i_2i_3}, \dots, a_{i_mi_1}$ vanishes. A system which is not a cycle system is called a chain system.*

We consider the n-dimensional $(n \geq 3)$ prey-predator loop system which satisfies (H) and is a special kind of cycle system, i.e., system (1) with an interaction matrix A_1 as follows:

$$A_1 = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 \\ 0 & a_{32} & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}.$$

If $a_{1n} = a_{n1} = 0$, matrix A_1 takes the form

$$A_2 = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 \\ 0 & a_{32} & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}.$$

According to Definition 2, it is easy to check that system (1) with interaction matrix A_2 is just a prey-predator chain system and matrix A_2 is Volterra-Liapunov semistable. In [1,2,9,10], it is proved that the locally asympototically stable equilibrium point x^* of system (1) with A_2 is globally stable for $n \leq 7$ which gives an affirmative answer to Chen's conjecture [7].

For $n = 3$, if matrix A_1 is not Volterra-Liapunov semistable, Roy and Solimano [15] showed that system (1) with it may possess a May-Leonard type heteroclinic cycle as an attractor. For general n, Radheffer and Zhou's theorems [8] imply that system (1) with matrix A_1 is globally stable provided A_1 is Volterra-Liapunov stable and $a_{ii}a_{i+1,i+1} \neq 0$, where $a_{n+1,n+1} = a_{11}$. In fact, in the latter case, the LaSalle's invariant set of the system is simply identical with $\{x^*\}$.

In this paper, we consider system (1) with the Volterra-Liapunov semistable matrix A_1 of which except one, without loss of generality, say a_{11}, all diagonal elements are zero. Then the matrix has the form

$$A_3 = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & 0 & 0 \\ 0 & a_{32} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n} \\ a_{n1} & 0 & 0 & \cdots & a_{n,n-1} & 0 \end{pmatrix}.$$

Unlike the results obtained by Redheffer and Zhou in [8], system (1), in some cases, with A_3 has its LaSalle's invariant set composed of nonconstant periodic solutions. In next section we will show that in lower dimensional cases, if A_3 is Volterra-Liapunov semistable, then system (1) with A_3 has the property that local symptotic stability of x^* implies its global stability.

3. MAIN RESULTS

In this section, we will show that system (1) with A_3 is mostly globally stable for lower dimensional systems. Namely, we have the following theorem.

THEOREM 1. *For $n = 4$, if A_3 is Volterra-Liapunov semistable and x^* is locally asymptotically stable, then x^* is globally stable.*

THEOREM 2. *For $n = 3$ or 5, if A_3 is Volterra-Liapunov semistable, then x^* is globally stable.*

Based on the above theorems, we propose the following.

CONJECTURE. *If A_3 is Volterra-Liapunov semistable, then system (1) with A_3 is globally stable for*

(i) *$n =$ odd;*
(ii) *$n =$ even, provided x^* is locally asymptotically stable.*

In what follows, suppose, without loss of generality, that $x_i^* = 1$ for $i = 1, \dots, n$. In the case of $n = 4$, system (1) with matrix A_3 reads:

$$\begin{aligned} \dot{x}_1 &= x_1 \left[a_{11}(x_1 - 1) + a_{12}(x_2 - 1) + a_{14}(x_4 - 1)\right], \\ \dot{x}_2 &= x_2 \left[a_{21}(x_1 - 1) + a_{23}(x_3 - 1)\right], \\ \dot{x}_3 &= x_3 \left[a_{32}(x_2 - 1) + a_{34}(x_4 - 1)\right], \\ \dot{x}_4 &= x_4 \left[a_{41}(x_1 - 1) + a_{43}(x_3 - 1)\right]. \end{aligned} \tag{3}$$

By Lemma 2, it is known that the LaSalle's invariant set of the above system is composed of the solutions of the following system [14]:

$$\dot{x}_2 = x_2 \left[a_{23}(x_3 - 1)\right], \tag{4}$$
$$\dot{x}_3 = x_3 \left[a_{32}(x_2 - 1) + a_{34}(x_4 - 1)\right], \tag{5}$$
$$\dot{x}_4 = x_4 \left[a_{43}(x_3 - 1)\right], \tag{6}$$
$$a_{12}(x_2 - 1) + a_{14}(x_4 - 1) = 0. \tag{7}$$

LEMMA 4. *System (4)–(7) possesses nonconstant solutions if and only if*

$$a_{12} + a_{14} = 0, \qquad a_{23} = a_{43} \qquad \text{and} \qquad a_{23}(a_{32} + a_{34}) < 0. \tag{8}$$

PROOF.

NECESSITY. Differentiating (7) along the nonconstant solution $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ and substituting (4),(6) into it, by removing a factor $x_3 - 1$, we obtain

$$a_{12}a_{23}x_2 + a_{14}a_{43}x_4 = 0. \tag{9}$$

Note that here we used the fact that $x_3 - 1 \neq 0$. Since $x_3 - 1 = 0$ implies that $\dot{x}_2 = \dot{x}_4 = 0$ from system (4)–(7), in this case, $x_2 = x_4 = c > 0$, where c is a constant. Clearly, $c = 1$ by the uniqueness of the positive equilibrium of system (3). Differentiating (9) once and following the same procedure as above, we have

$$a_{12}a_{23}^2 x_2 + a_{14}a_{43}^2 x_4 = 0. \tag{10}$$

Clearly, (9) and (10) imply $a_{23} = a_{43}$ which together with (7) and (10) leads to $a_{12} + a_{14} = 0$. In this case, system (4)–(7) is equivalent to the following system:

$$\dot{x}_2 = x_2 \left[a_{23} \left(x_3 - 1\right)\right], \tag{11}$$

$$\dot{x}_3 = x_3 \left[\left(a_{32} + a_{34}\right) \left(x_2 - 1\right)\right]. \tag{12}$$

By Theorem 18.2 in [16], system (11)–(12) has bounded nonconstant solution which does not tend to $(1,1)$ if and only if $a_{23}(a_{32} + a_{34}) < 0$.

SUFFICIENCY. According to the condition (8), the solutions of system (11)–(12) are ones of (4)–(7). Since (11)–(12) has nonconstant periodic solutions, then (4)–(7) has also nonconstant periodic solutions.

This completes the proof of the lemma.

PROOF OF THEOREM 1. By Lemma 4 and Theorem 18.2 in [16], whenever system (4)–(7), namely, the LaSalle's invariant set of system (3), possesses nonconstant solutions, every neighborhood of x^* contains nonconstant solutions. In this case, the positive equilibrium point x^* is not locally, asymptotically stable. Therefore, when x^* is locally, asymptotically stable, the LaSalle's invariant set of system (3) must not contain any nonconstant solutions; that is, system (4)–(7) only has solution x^*. Hence, the feasible equilibrium point x^* must be globally stable.

This completes the proof of the theorem.

Since Theorem 2 is easy to prove for $n = 3$, we only prove the case of $n = 5$. In this case, system (1) with A_3 reads

$$\begin{aligned}
\dot{x}_1 &= x_1 \left[a_{11} \left(x_1 - 1\right) + a_{12} \left(x_2 - 1\right) + a_{15} \left(x_5 - 1\right)\right], \\
\dot{x}_2 &= x_2 \left[a_{21} \left(x_1 - 1\right) + a_{23} \left(x_3 - 1\right)\right], \\
\dot{x}_3 &= x_3 \left[a_{32} \left(x_2 - 1\right) + a_{34} \left(x_4 - 1\right)\right], \\
\dot{x}_4 &= x_4 \left[a_{43} \left(x_3 - 1\right) + a_{45} \left(x_5 - 1\right)\right], \\
\dot{x}_5 &= x_5 \left[a_{51} \left(x_1 - 1\right) + a_{54} \left(x_4 - 1\right)\right].
\end{aligned} \tag{13}$$

By Lemma 2, it is known that the LaSalle's invariant set of the system (13) is composed of the solutions of the following system:

$$\dot{x}_2 = x_2 \left[a_{23} \left(x_3 - 1\right)\right], \tag{14}$$

$$\dot{x}_3 = x_3 \left[a_{32} \left(x_2 - 1\right) + a_{34} \left(x_4 - 1\right)\right], \tag{15}$$

$$\dot{x}_4 = x_4 \left[a_{43} \left(x_3 - 1\right) + a_{45} \left(x_5 - 1\right)\right], \tag{16}$$

$$\dot{x}_5 = x_5 \left[a_{54} \left(x_4 - 1\right)\right], \tag{17}$$

$$a_{12} \left(x_2 - 1\right) + a_{15} \left(x_5 - 1\right) = 0. \tag{18}$$

In the next section, we will show that if system (14)–(18) possesses a nonconstant solution, then $a_{12} + a_{15} = 0$.

LEMMA 5. *If $a_{12} + a_{15} = 0$, then system (14)–(18) has only solution $x = (1, 1, 1, 1)$.*

PROOF. $a_{12}+a_{15}=0$ together with system (18) implies $x_2 = x_5$. Furthermore, by (14) and (17), we have $a_{23}(x_3-1) - a_{54}(x_4-1) = 0$. Substituting $(x_4-1) = a_{23}(x_3-1)/a_{54}$ into (15), then (14) and (15) lead to

$$\begin{aligned} \dot{x}_2 &= x_2\left[a_{23}\left(x_3-1\right)\right], \\ \dot{x}_3 &= x_3\left[a_{32}\left(x_2-1\right)+\frac{a_{23}a_{34}}{a_{54}}\left(x_3-1\right)\right]. \end{aligned} \tag{19}$$

By Lemma 3 and Theorem 18.2 in [16], system (19) has only one solution $(x_2, x_3) = (1, 1)$. Hence, system (14)–(18) has only solution $x = (1, 1, 1, 1)$. This proves the lemma.

4. WU'S WELL-ORDERING PRINCIPLE

To prove the main result, we use the principle of characteristic sets which was introduced by Ritt [17] in the context of his work on differential algebra and has been considerably developed by Wu [3]. The great success of theorem proving has stimulated the renewed interest in the characteristic sets method. To limit the space, we give here only the basic principle, i.e., the well-ordering principle, and illustrate how this principle together with some computational techniques as well as computer algebra system MATHEMATICA can be applied to determine the conditions for system (14)–(18) to contain nonconstant solutions.

Let (PS) $= \{f_1(x_1,\dots,x_n),\dots,f_s(x_1,\dots,x_n)\}$ be any finite set of polynomials in n ordered variables $x_1 \prec \cdots \prec x_n$ with coefficients in certain basic field of characteristic 0, for instance, the rational field Q. We designate the totality of zeros of polynomials of (PS) by Zero(PS). If G is any other nonzero polynomial, the subset of Zero(PS) for which $G \neq 0$ will be denoted as Zero(PS$/G$). The following is the basic principle of the characteristic sets method [3,4].

LEMMA 6. *(Wu's well-ordering principle.) Given a set of polynomials* (PS), *one can compute by an algorithmic method another set of polynomials* (CS), *called the characteristic set of* (PS), *of the triangular form*

$$\text{(CS)} \qquad \begin{aligned} &c_1\left(u_1,\dots,u_d;y_1\right), \\ &c_2\left(u_1,\dots,u_d;y_1,y_2\right), \\ &\dots, \\ &c_r\left(u_1,\dots,u_d;y_1,\dots,y_r\right), \end{aligned}$$

so that

$$\text{Zero}\left(\frac{\text{CS}}{J}\right) \subset \text{Zero(PS)} \subset \text{Zero(CS)}, \tag{20}$$

$$\text{Zero(PS)} = \text{Zero}\left(\frac{\text{CS}}{J}\right) \cup \bigcup_i \text{Zero}\left(\text{PS}_i\right), \tag{21}$$

where $u_1,\dots,u_d; y_1,\dots,y_r (d+r=n)$ is a rearrangement of $x_1,\dots,x_n$, $J = \prod_i I_i$, I_i is the leading coefficient of c_i as polynomial in y_i, called the initial of c_i, and $(\text{PS}_i) = (\text{PS}) \cup \{I_i\}$.

The algorithm for triangulating the polynomial set proceeds basically the successive pseudo-division of polynomials in certain manner. From (20) and (21) the relation between the zeros of (PS) and (CS) is clear. That is, any zero of (PS) is a zero of (CS) and, conversely, any zero of (CS) which does not make the vanishing of the initials of polynomials in (CS) is also a zero of (PS). Therefore, under the condition that all initials are not equal to 0, both (PS) and (CS) have same zero set. For those zeros of (PS) making the vanishing of some initial I_i, we may consider for the the enlarged polynomial set (PS_i) obtained from (PS) by adjoining I_i to it as required. Furthermore, in proceeding with each (PS_i) as (PS) by the same principle we would get finally a zero decomposition of the form

$$\mathrm{Zero(PS)} = \bigcup_i \mathrm{Zero}\left(\frac{\mathrm{CS}_i}{J_i}\right), \tag{22}$$

in which (CS_i) is of triangular form as (CS) and J_i is the product of initials of polynomials in (CS_i) for each i.

Now, we consider the problem about the determination for LaSalle's invariant set of system (13), i.e., system (14)–(18) to have nonconstant solutions. For this purpose, let

$$V0 = a_{12}(x_2 - 1) + a_{15}(x_5 - 1).$$

Then the successive derivatives Vi of order i, $i = 1, 2, 3$, of $V0$ with respect to t are $V1 = V1(a_{ij}; x_2, x_3, x_4, x_5)$, $V2 = V2(a_{ij}; x_2, x_3, x_4, x_5)$ and $V3 = V3(a_{ij}; x_2, x_3, x_4, x_5)$. Hence, we obtain a polynomial set (PS) $= \{V0, V1, V2, V3\}$ of a_{ij} and x_k, $k = 2, 3, 4, 5$, where $V0$, $V1$, $V2$ and $V3$ consist of 4, 4, 14 and 72 terms, respectively.

Theoretically, according to Wu's well-ordering principle (Lemma 6), we can well order the polynomial set (PS) and obtain the corresponding (CS) with the first polynomial involving variables a_{ij} and, for example, x_4: $V_f(a_{ij}; x_4)$. Since the polynomial set (PS) is zero on the set of nonconstant solutions of system (14)–(18), by the zero relation (20) and (21), $V_f(a_{ij}; x_4)$ is also zero along the nonconstant solutions of (14)–(18). This means $V_f(a_{ij}; x_4) = 0$ must be an identical equation for x_4; that is, the coefficients of similar terms of V_f must be zero. That all the coefficients of V_f are zero gives us a set of polynomial equations of a_{ij} from which we can reach the result. But owing to the complexity, it seems impossible to obtain the corresponding (CS) from (PS), directly. According to the well-ordering principle and the zero relations (20) and (21), we know that to obtain (CS) is equivalent to get the coefficient polynomials of variants in (CS). The idea to solve our problem is to repeatedly use Wu's well ordering principle and obtain different characteristic sets from which we can find useful polynomials of a_{ij}. The main result will be derived based on this idea.

5. THE PROOF OF THEOREM 2

In this section, we give the derivation of Theorem 2 in detail, by using the computer algebra system MATHEMATICA, on the basis of the characteristic sets method.

PROOF OF THEOREM 2. To prove Theorem 2, by Lemma 5, it is enough to show that if the system (14)–(18) has a nonconstant solution, then $a_{12} + a_{15} = 0$. Throughout this section, by supposing that system (14)–(18) possesses a nonconstant solution $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$ with $x_1(t) = 1$ and $a_{12} + a_{15} \neq 0$, we will get a contradiction.

Let

$$V0 = a_{12}(x_2 - 1) + a_{15}(x_5 - 1). \tag{23}$$

Then the successive derivatives, Vi of order i of $V0$, give us the polynomial set (PS) $= \{V0, V1, V2, V3\}$ of a_{ij} and x_k. Now, we prove Theorem 2 in five steps.

STEP 1. We show that

$$(a_{43} - a_{23})^2 + (a_{54} - a_{34})^2 \neq 0. \tag{24}$$

Since $x_5 \neq 0$, from $V0 = 0$, we have

$$x_2 \neq \frac{(a_{12} + a_{15})}{a_{12}}. \tag{25}$$

By taking x_5 as a function of x_2 and a_{ij} from $V0 = 0$, substituting it into $V1$, we can obtain its numerator $V11(a_{ij}; x_2, x_3, x_4)$ with eight terms. Solving x_4 from $V11 = 0$ as a function of a_{ij}, x_2

and x_3, then substituting $x_4 = x_4(a_{ij}; x_2, x_3)$, $x_5 = x_5(a_{ij}; x_2)$, $a_{43} = a_{23}$ and $a_{54} = a_{34}$ into $V2$ and $V3$, respectively, and taking out the numerators of the resulting expressions, we obtain two polynomials $P2(a_{ij}; x_2, x_3)$ with a factor a_{34} removed and $P3(a_{ij}; x_2, x_3)$ with a factor a_{34}^2 removed with 51 and 180 terms, respectively. The resultant of $P2$ and $P3$ with respect to x_3 gives us the polynomial $PP(a_{ij}; x_2)$ with 1055 terms.

Now we choose some coefficient polynomials of x_2^n in $PP(a_{ij}; x_2)$ denoted by PPn as follows:

$$PP11 = a_{12}^8 a_{15} a_{23}^2 (a_{23} - a_{34}) a_{45} (-a_{15}a_{32} + a_{12}a_{45})^2 = 0$$

together with $PP10$, $PP9$, $PP3$, $PP2$ and $PP1$. Clearly, $PP11 = 0$ implies

$$a_{23} - a_{34} = 0, \tag{26}$$

or

$$-a_{15}a_{32} + a_{12}a_{45} = 0. \tag{27}$$

$PP10$ together with (26) (or $PP9$ together with (27)) implying (27) (or (26)) means that (26) and (27) must hold simultaneously.

Simplified by (26), (27) and by removing some nonzero factors, $PP1 = 0$, $PP2 = 0$ and $PP3 = 0$ lead to

$$7a_{12}a_{23} + a_{15}a_{23} + 7a_{12}a_{32} + 3a_{15}a_{32} = 0, \tag{28}$$

$$19a_{12}^2a_{23} + 6a_{12}a_{15}a_{23} + 922a_{12}^2a_{32} + 17a_{12}a_{15}a_{32} + 2a_{15}^2a_{32} = 0, \tag{29}$$

$$25a_{12}^2a_{23} + 13a_{12}a_{15}a_{23} + 38a_{12}^2a_{32} + 41a_{12}a_{15}a_{32} + 9a_{15}^2a_{32} = 0. \tag{30}$$

By eliminating a_{15}, a_{23} and a_{32} from (28), (29) and (30), we obtain

$$-6496392a_{12}^2 = 0,$$

which contradicts to $a_{ij} \neq 0$. Hence (24) holds true.

STEP 2. We show that

$$-a_{15}a_{43} + a_{12}a_{45} = 0 \quad \text{and} \quad -a_{12}a_{34} + a_{15}a_{32} = 0. \tag{31}$$

Note that system (14)–(18) has a symmetric form so that if we use the replacements

$$(x_2, x_3, x_4, x_5) \to (x_5, x_4, x_3, x_2)$$

and

$$(a_{12}, a_{23}, a_{32}, a_{34}, a_{43}, a_{45}, a_{54}, a_{15}) \to (a_{15}, a_{54}, a_{43}, a_{34}, a_{32}, a_{23}, a_{12}),$$

the system will be unchanged. Therefore, we only need to prove one of equalities in (31).

Similar to Step 1, substituting $x_5 = x_5(a_{ij}; x_2)$ solved from $V0 = 0$ into $V1$, by solving the new $V1 = 0$, we obtain $x_4 = x_4(a_{ij}; x_2, x_3)$. Substituting the obtained $x_4 = x_4(a_{ij}; x_2, x_3)$ and $x_5 = x_5(a_{ij}; x_2)$ into $V2$ and $V3$ and simplifying them, we obtain two polynomials $Q2$ and $Q3$ with 53 and 342 terms, respectively.

Now we write $Q2$ and $Q3$ as polynomials of x_3 such that the coefficients polynomials of them are ones of a_{ij} and x_2. By calculating the resultant QQ of $Q2$ and $Q3$ with respect to x_3, the obtained QQ is a summation of some products of polynomials with variables x_2 and a_{ij}. It seems impossible to expand QQ as a polynomial of x_2; we now get the coefficient polynomial denoted by $QQ1$ of x_2 in QQ by letting $x_2 = 0$ in the derivative of QQ.

$$QQ1 = a_{12}a_{15}^6 (a_{12} + a_{15})^{11} a_{23}^2 a_{32} (a_{23} - a_{43}) a_{43}^3 (-a_{15}a_{43} + a_{12}a_{45}) a_{54}^{11}.$$

Since $a_{12}+a_{15}\neq 0$, $QQ1=0$ implies

$$(-a_{15}a_{43}+a_{12}a_{45})\,(a_{23}-a_{43})=0. \tag{32}$$

By symmetricity, we have

$$(a_{15}a_{32}-a_{12}a_{34})\,(a_{34}-a_{54})=0. \tag{33}$$

Obviously, (32) and (33) together with Step 1 and the symmetricity of system (14)–(18) imply (31).

STEP 3. Now we prove that if (31) holds, then

$$a_{43}\neq a_{23} \quad \text{and} \quad a_{54}\neq a_{34}. \tag{34}$$

We assume, according to Step 1, that $a_{43}=a_{23}$ but $a_{54}\neq a_{34}$ to obtain a contradiction.

By eliminating x_3, x_4 and x_5, we obtain a polynomial in x_2 with 2160 terms. By factorizing this polynomial and removing the nonzero factor

$$a_{23}^2a_{32}^2a_{54}^3\,(-a_{34}+a_{54})\,(-1+x_2)\,x_2^3\,(a_{32}+a_{34}-a_{32}x_2)^3$$

from it, we obtain a 740 terms' polynomial RR. Note that in the procedure of obtaining these polynomials, the following three relations from (31) and the assumption are used:

$$\begin{aligned} a_{15}&=\frac{a_{12}a_{34}}{a_{32}},\\ a_{45}&=\frac{a_{15}a_{43}}{a_{12}},\\ a_{43}&=a_{23}. \end{aligned}$$

Now we choose coefficient polynomials RRn of x_2^n in RR for $n=0,9$ and 10.

$$RR10=a_{23}a_{32}^8\,(a_{23}+a_{32})^2\,a_{54}^2\,(-a_{23}+a_{54})\,(-a_{34}+a_{54})=0.$$

Since $a_{54}\neq a_{34}$, we have

$$a_{23}+a_{32}=0 \tag{35}$$

or

$$a_{54}-a_{23}=0. \tag{36}$$

It is easy to check that (35) and $RR0=0$ imply (36). Now simplified by $a_{54}=a_{23}$, $RR9=0$ and $RR0=0$ lead to

$$(a_{23}+a_{32})\,(-a_{23}+a_{34})\,(a_{32}+a_{34})=0, \tag{37}$$

$$(a_{23}-a_{32})\,(a_{32}+a_{34})=0, \tag{38}$$

respectively. Clearly, $a_{15}+a_{12}\neq 0$ and (31) imply $a_{34}+a_{32}\neq 0$. Therefore, (37) and (38) lead to

$$\begin{aligned} (a_{23}+a_{32})\,(-a_{23}+a_{34})&=0,\\ a_{23}-a_{32}&=0, \end{aligned}$$

which imply $a_{34}=a_{23}$. This and (36) contradict $a_{54}\neq a_{34}$.

STEP 4. We will show that

$$-a_{23}a_{34}+a_{43}a_{54}=0. \tag{39}$$

By a similar procedure to Steps 1 and 2, simplified by (31), $V0 = 0$ and $V1 = 0$, $V2$ and $V3$ lead to $S2$ and $S3$ with 59 and 398 terms, respectively.

Let $S2$ and $S3$ be two polynomials of x_4 with each coefficient polynomial as a function of a_{ij} and x_3. Replacing each coefficient polynomial of $R2$ and $R3$ by the corresponding polynomial with maximal degree, we get two *new* polynomials $\bar{S2}$ and $\bar{S3}$ by removing some nonzero factors. $\bar{S2}$ and $\bar{S3}$ have 8 and 17 terms, respectively. As in the proof of Step 3, the resultant of $\bar{S2}$ and $\bar{S3}$ with respect to x_3 gives us a polynomial of x_2 with maximal degree 13. The coefficient x_2^{13} being zero leads to

$$a_{23}^7 a_{43} \left(-a_{23} + a_{43}\right)^2 (a_{34} - a_{54})\, a_{54}^5 \left(-a_{23}a_{34} + a_{43}a_{54}\right)^3 = 0.$$

By Step 3, we obtain (39).

From the above steps, we have shown that if system (14)–(18) has a nonconstant solution and $a_{12} + a_{15} \neq 0$, then (31), (34) and (39) must hold. In the last step we will show that in fact $a_{12} + a_{15} = 0$, if system (14)–(18) has a nonconstant solution.

STEP 5. In this step, under the conditions (31), (34) and (39), we show that

$$a_{12} + a_{15} = 0. \tag{40}$$

Eliminating x_3, x_4 and x_5 from (PS) with a simplification of (31) and (39), we obtain a polynomial in a_{ij} and x_2 with 2080 terms. Factoring this polynomial and removing a nonzero factor

$$a_{12}^2 a_{32}(-1 + x_2)x_2^2(-a_{12} - a_{15} + a_{12}x_2)^2$$

lead to a 1102 terms' polynomial TT in a_{ij} and x_2. Now, we denote the coefficient polynomial of x_2^n as TTn. To obtain (40), $TT0$, $TT1$, $TT2$, $TT10$ and $TT11$ are chosen.

First, $TT0 = 0$ implies

$$a_{32} = a_{23}, \tag{41}$$

or

$$a_{32} = -\frac{a_{12}^2 a_{23}}{(a_{12}a_{15} + a_{15}^2)}. \tag{42}$$

CASE 5.1. Simplified by (41), $TT11 = 0$ leads to

$$a_{23} + a_{43} = 0, \tag{43}$$

or

$$a_{15}a_{23} + a_{12}a_{43} = 0. \tag{44}$$

Simplified by (41) and (43), $TT10 = 0$ leads to (simplified by (41) and (44), $TT10 = 0$ also leads to)

$$a_{15} = a_{12}. \tag{45}$$

Simplified by (41), (43) and (45), $TT1 = 0$ leads to

$$1024 a_{12}^{12} a_{23}^6 = 0,$$

which contradicts to $a_{ij} \neq 0$.

CASE 5.2. Simplified by (42), $TT1 = 0$ leads to

$$a_{43} = \frac{2a_{12}a_{23}}{(a_{12} + a_{15})}, \tag{46}$$

since $a_{43} \neq a_{23}$, by Step 3. Equation (46) and $TT2 = 0$ imply

$$a_{15} = a_{12}. \tag{47}$$

But (46) and (47) contradict $a_{43} \neq a_{23}$.

Hence, the above steps show that if system (14)–(18) has a nonconstant solution, then $a_{12} + a_{15} = 0$. By Lemma 5, the theorem is proved.

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